1.	**D**
2.	**A**
3.	**B**
4.	**B**
5.	**A**
6.	**E**
7.	**B**
8.	**D**
9.	**B**
10.	**D**
11.	**C**
12.	**A**
13.	**E**
14.	**E**
15.	**C**
16.	**A**
17.	**C**
18.	**D**
19.	**A**
20.	**E**
21.	**C**
22.	**B**
23.	**D**
24.	**C**
25.	**C**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**
supported by
Institute and Faculty of Actuaries

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 5 NOVEMBER 2015

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' at the end of a solution.

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

The UKMT is a registered charity.

1. **D** The expression $2015^2 - 2016 \times 2014$ can be written as $2015^2 - (2015+1)(2015-1)$ which simplifies, using the difference of two squares, to $2015^2 - (2015^2 - 1) = 1$.

2. **A** Rearranging $6x = \dfrac{150}{x}$ gives $x^2 = \dfrac{150}{6}$, so $x^2 = 25$. This has two solutions, $x = 5$ and $x = -5$. Therefore the sum of the solutions is $5 + (-5) = 0$.

3. **B** When 50 litres of petrol cost £40, 1 litre cost $\dfrac{£40}{50}$ which is 80 pence. More recently, 1 litre cost $\dfrac{£50}{40} = 125$ pence. The percentage increase is then $\dfrac{\text{actual increase}}{\text{original price}} \times 100$ which is $\dfrac{45}{80} \times 100 = \dfrac{450}{8} = 56.25$. So the approximate increase is 56%.

4. **B** Let the radius of the smaller circle be r and so the radius of the larger circle is $2r$. The area of the smaller circle is then πr^2 and the area of the larger circle is $\pi \times (2r)^2$ which is $4\pi r^2$. The fraction of the larger circle which is outside the smaller circle is then $\dfrac{4\pi r^2 - \pi r^2}{4\pi r^2} = \dfrac{3\pi r^2}{4\pi r^2} = \dfrac{3}{4}$.

5. **A** The mean of 17, 23 and $2n$ is given to be n, so $\dfrac{17 + 23 + 2n}{3} = n$ which gives $40 + 2n = 3n$. As n is then 40, the sum of the digits of n is 4.

6. **E** The prime numbers which are the sums of pairs of numbers in touching circles are all odd as they are greater than 2. This means that any two adjacent circles in the diagram must be filled with one odd number and one even number. The number 10 may not be placed on either side of 5, since $10 + 5 = 15 = 3 \times 5$. So either side of the 5 must be 6 and 8. Below 6 and 8 must be 7 and 9 respectively leaving 10 to be placed in the shaded circle at the bottom.

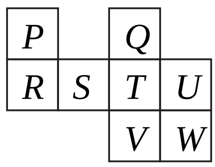
7. **B** Evaluating each option gives

A $\dfrac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)} = \dfrac{1}{2} \times \dfrac{4}{3} = \dfrac{2}{3}$ B $\dfrac{1}{\left(\frac{\left(\frac{2}{3}\right)}{4}\right)} = \dfrac{1}{\left(\frac{2}{12}\right)} = \dfrac{12}{2} = 6$ C $\dfrac{\frac{\left(\frac{1}{2}\right)}{3}}{4} = \dfrac{\left(\frac{1}{6}\right)}{4} = \dfrac{1}{24}$

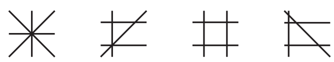
D $\dfrac{1}{\frac{2}{\left(\frac{3}{4}\right)}} = \dfrac{1}{\left(\frac{8}{3}\right)} = \dfrac{3}{8}$ E $\dfrac{\left(\frac{1}{\left(\frac{2}{3}\right)}\right)}{4} = \dfrac{\left(\frac{3}{2}\right)}{4} = \dfrac{3}{8}$.

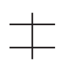
So B has the largest answer.

8. **D** Let the squares in the diagram be labelled as shown. Each of the nets formed from six squares must contain all of R, S and T. The net must also include one of P and Q (but not both as they will fold into the same position), and any two of U, V and W. This therefore gives $2 \times 3 = 6$ different ways.



9. **B** Possible configurations of four different straight lines drawn in a plane are shown here to give 1, 3, 4 and 5 points of intersection respectively. In order to have exactly 2 points of intersection, two of the straight lines would need to lie in the same position and so would not be 'different'.





10. **D** The total of the numbers from 1 to 20 is $\frac{1}{2} \times 20 \times (20+1) = 210$. If Milly and Billy have totals which are equal, their totals must each be 105. Milly's total, of the numbers from 1 to n, is $\frac{1}{2}n(n+1)$ so $\frac{1}{2}n(n+1) = 105$ which gives $n^2 + n = 210$. Therefore $n^2 + n - 210 = 0$ which factorises to give $(n + 15)(n - 14) = 0$. As n is a positive integer, $n = 14$.

11. **C** There are several different ways to count systematically the number of towers that Rahid can build. Here is one way.

	All blocks the same size			Exactly two blocks the same size						All blocks of different sizes
	10 10 10	6 6 6	4 4 4	4 10 10	6 10 10	4 6 6	10 6 6	6 4 4	10 4 4	4 6 10
Total height	30	18	12	24	26	16	22	14	18	20

So there are nine different heights of tower (as the height of 18cm can be made from $6 + 6 + 6$ or $10 + 4 + 4$).

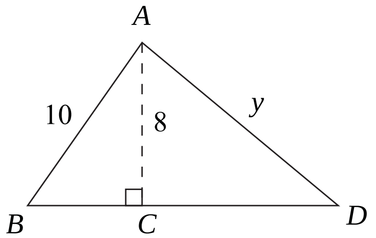
12. **A** Each of the three sides of triangle PQR is a tangent to the circle. Two tangents to a circle which meet at a point are of equal length. So QU and QS are of equal length. Similarly $RT = RS$. This means that $\angle QUS = \angle QSU = \frac{1}{2}(180 - \alpha)$ and also $\angle RTS = \angle RST = \frac{1}{2}(180 - \beta)$. At S we can consider the sum of the three angles, so $\frac{1}{2}(180 - \alpha) + \gamma + \frac{1}{2}(180 - \beta) = 180$. Simplifying gives $90 - \frac{1}{2}\alpha + \gamma + 90 - \frac{1}{2}\beta = 180$ and so $\gamma = \frac{1}{2}(\alpha + \beta)$.

13. **E**

Knave of	Mon	Tue	Wed	Thu	Fri	Sat	Sun	Mon
Hearts	T	T	T	T	L	L	L	T
Diamonds	T	L	L	L	T	T	T	T

When a knave says "Yesterday I told lies" it could be that today he is telling the truth and he did indeed tell lies yesterday. In the table, this is a T preceded by an L.
It could also be that today he is lying, in which case he was in fact telling the truth yesterday. In the table, this is an L preceded by a T. The only day when one or the other of these options applies to each knave is Friday.

14. **E**



Let the vertices of the triangle be labelled A, B and D as shown. Let the point where the perpendicular from A meets BD be labelled C. The area of triangle ABD is given as 88. As BD is 22, AC must be 8. Considering triangle ABC and using Pythagoras' Theorem gives $BC = 6$. The remainder of the base CD is then $22 - 6 = 16$. Considering triangle ACD and using Pythagoras' Theorem again gives $y^2 = 8^2 + 16^2 = 8^2(1^2 + 2^2) = 8^2 \times 5$. So $y = 8\sqrt{5}$.

15. **C** Let the original water level in the larger vase be h cm. The volume of water at the start is then $\pi \times 10^2 \times h$ cm^3. The volume of water completely within the vase is constant, but when the smaller vase is pushed down, some of the water moves into it. In the end the depth of the water in the larger vase is the same as the height of the smaller vase itself, which is 16 cm. We are given that the final depth of water in the smaller vase is 8 cm. So the total volume of water is then $\pi \times 10^2 \times 16$ cm^3 less the gap in the top half of the smaller vase. So $\pi \times 10^2 \times h = \pi \times 10^2 \times 16 - \pi \times 5^2 \times 8$, giving $100\pi h = 1600\pi - 200\pi$ and therefore $h = 14$.

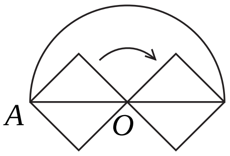
16. **A** Let the six Fnargs in their final positions be denoted by $F_1F_2F_3F_4F_5F_6$. There are six choices for F_1. Once this Fnarg is chosen, the colours of the Fnargs must alternate all along the line and so we need only consider the number of heads. There are $3 - 1 = 2$ choices for F_2 as the number of heads for $F_2 \neq$ the number of heads for F_1. There is only one choice for F_3 as F_3 cannot have the same number of heads as F_2 or F_1 (F_3 and F_1 are the same colour and so have different numbers of heads). There is only one choice for F_4 as it is completely determined by F_3 and F_2, just as F_3 was completely determined by F_2 and F_1. There is only one choice for each of F_5 and F_6 as they are the last of each colour of Fnargs. The total number of ways of lining up the Fnargs is $6 \times 2 \times 1 \times 1 \times 1 \times 1$ which is 12.

17. **C** Let the radius of each of the smaller circles be r and let the centres of the circles be A, B, C and D in order. We are given that $ABCD$ is a square. When two circles touch externally, the distance between their centres equals the sum of their radii. Hence AB and BC have length $r + 1$ and AC has length $1 + 1 = 2$. By Pythagoras' Theorem $(r+1)^2 + (r+1)^2 = 2^2$, so $2(r+1)^2 = 2^2 = 4$ and therefore $(r+1)^2 = 2$. Square rooting both sides gives $r + 1 = \sqrt{2}$, as we must take the positive root, and so $r = \sqrt{2} - 1$.

18. **D** Expressed as a product of its prime factors, 10! is $2 \times 5 \times 3 \times 3 \times 2 \times 2 \times 2 \times 7 \times 2 \times 3 \times 5 \times 2 \times 2 \times 3 \times 2$ which is $2^8 \times 3^4 \times 5^2 \times 7$. This can be written as $(2^4 \times 3^2 \times 5)^2 \times 7$ so the largest integer k such that k^2 is a factor of 10! is $2^4 \times 3^2 \times 5$ which is 720.

19. **A** Let the length of the side of the smallest square be x cm. So the three squares have sides of lengths x cm, $(x + 8)$ cm and 50 cm respectively. The gradient of PQ is then $\frac{8}{x}$ and the gradient of PR is $\dfrac{50 - x}{x + x + 8}$. As P, Q and R lie on a straight line, $\dfrac{8}{x} = \dfrac{50 - x}{2x + 8}$ so $8(2x + 8) = x(50 - x)$. Expanding gives $16x + 64 = 50x - x^2$ and therefore $x^2 - 34x + 64 = 0$, giving $x = 2$ or 32.

20. **E**



Let the corner of the square about which it is rotated be O and the opposite vertex of the square be A. As the circle is rotated through 180° about O, the vertex A travels along a semicircle whose centre is O. The area coloured black by the ink is then formed from two half squares and a semicircle. The square has side-length 1, so $OA = \sqrt{2}$. The total area of the two half squares and the semicircle is $2 \times (\frac{1}{2} \times 1 \times 1) + \frac{1}{2} \times \pi \times (\sqrt{2})^2$ which is $1 + \pi$.

21. **C** All of the triangles in the diagram are similar as they contain the same angles. The sides of each triangle are therefore in the ratio $2 : 3 : 4$. First consider triangle APM. Let $AP = x$, so that $AM = 2x$. Now considering triangle TBM, as $BT = x$, $BM = \frac{4x}{3}$. The quadrilateral $AMSX$ is a parallelogram as AM is parallel to XS and MS is parallel to AX. So $AM = XS = 2x$. Similarly $QZ = BM = \frac{4x}{3}$. Considering the base of triangle XYZ, $XS + SQ + QZ = 4$. So $2x + x + \frac{4x}{3} = 4$ and therefore $x = \frac{12}{13}$.

22. **B** $f(x) = x + \sqrt{x^2 + 1} + \dfrac{1}{x - \sqrt{x^2 + 1}} = \dfrac{\left(x + \sqrt{x^2 + 1}\right)\left(x - \sqrt{x^2 + 1}\right) + 1}{x - \sqrt{x^2 + 1}}$. The numerator is $x^2 - \left(\sqrt{x^2 + 1}\right)^2 + 1 = -1 + 1 = 0$. So $f(x) = 0$. Hence $f(2015) = 0$.

23. **D** Let a four-digit positive integer be expressed as $1000a + 100b + 10c + d$ where a, b, c and d are all different. In the 24 possible permutations of a, b, c and d, each of the four letters appears in each position six times. Adding all 24 numbers together gives $1000(6a + 6b + 6c + 6d) + 100(6a + 6b + 6c + 6d) + 10(6a + 6b + 6c + 6d) + 6a + 6b + 6c + 6d$. The total is therefore $1111 \times 6(a + b + c + d)$ which factorises to $2 \times 3 \times 11 \times 101(a + b + c + d)$. As $a + b + c + d < 101$, the largest prime factor of the sum is 101.

24. **C** There are five cards in Peter's set that are printed with an integer that has no prime factors in common with any other number from 1 to 25. The five numbers are 1 (which has no prime factors) and the primes 13, 17, 19 and 23. These cards cannot be placed anywhere in the row of N cards. One possible row is: 11, 22, 18, 16, 12, 10, 8, 6, 4, 2, 24, 3, 9, 21, 7, 14, 20, 25, 15, 5. So the longest row is of 20 cards.

25. **C** Repeatedly using the rule that $f(xy) = f(x) + f(y)$ allows us to write $f(500)$ as $f(2 \times 2 \times 5 \times 5 \times 5) = f(2) + f(2) + f(5) + f(5) + f(5) = 2f(2) + 3f(5)$. We are given values for $f(40)$ and $f(10)$ and from them we need to calculate the values of $f(2)$ and $f(5)$. Now $f(40)$ can be written as $f(2) + f(2) + f(10)$ so $20 = 2f(2) + 14$ and therefore $f(2) = 3$. Similarly $f(10) = f(2) + f(5)$ so $14 = 3 + f(5)$ giving $f(5) = 11$. So $f(500) = 2f(2) + 3f(5) = 2 \times 3 + 3 \times 11 = 39$.

For reasons of space, these solutions are necessarily brief. There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:
http://www.ukmt.org.uk/